Resit Exam Stochastic Processes

July 2020

- You have from 8.30 until 11.30. This includes the time needed to take pictures of your work and upload it to nestor dropbox.
- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- During the entire time you should be connected to the skype group call and be on camera, with sound turned on. Failure to do this will count as cheating.

Exercise 1 (20 pts)
Consider a Galton-Watson tree with offspring distribution given by $X \stackrel{d}{=} \text{Bi}(2, p)$. Determine the probability of extinction q.

Exercise 2 (20 pts)
Let $X_1, X_2, \ldots$ be i.i.d. Be(p)-distributed (i.e. $\mathbb{P}(X_1 = 1) = p, \mathbb{P}(X_1 = 0) = 1 - p$) and let $N \stackrel{d}{=} \text{Po}(\mu)$ be independent of $X_1, X_2, \ldots$.

Find the probability generating function of

$$Z := \sum_{i=1}^{N} X_i.$$

What distribution does Z follow?

Exercise 3 (20 pts)
Prove Stirling's formula $n! = (1 + o_n(1)) \cdot \sqrt{2\pi n} \cdot (n/e)^n$.

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Exercise 4 (a:10, b:10 pts)
Consider the sequence of random variables $X_1, X_2, \dots,$ taking values in $\{0, \dots, N\}$ with the property that

$$\mathbb{P}(X_{i+1} = j | X_1 = j_1, \dots, X_i = j_i) = \begin{cases} p & \text{if } j = j_i + 1 \text{ and } j_i < N, \\ 1-p & \text{if } j = j_i - 1 \text{ and } j_i > 0, \\ 1 & \text{if } j = j_i = 0 \text{ or } j = j_i = N, \\ 0 & \text{otherwise.} \end{cases}$$

a) Show that the sequence $Y_1, Y_2, \dots$ given by $Y_i = \left(\frac{1-p}{p}\right)^{X_i}$ is a martingale.

b) By the martingale convergence theorem, $Y_n \xrightarrow[n\to\infty]{\mathrm{d}} Y$ for some random variable Y. Give the pmf of Y in terms of the value of X_1 and p. That is, find an expression for $\mathbb{P}(Y = y | X_1 = x)$ as a function of y, x, p.

Exercise 5 (20 pts)
At a certain popular beach two species of man-eating sharks are known to occur. Starting from time $t = 0$, the number of attacks by hammerhead sharks is described by a Poisson process $\mathcal{P}_1$ of intensity μ_1, and the number of attacks by great white sharts is described by a Poisson process $\mathcal{P}_2$ of intensity μ_2. Since it concerns distinct species of sharks, we may assume the Poisson processes are independent. We can furthermore assume that at this particular beach no other species of man-eating sharks occur. Show that the probability that the first attack is by a hammerhead shark equals $\frac{\mu_1}{\mu_1+\mu_2}$.

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